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Harmonic Analysis in Coding Theory

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Abstract

This is a reading report of [Bac10]. This report synthesizes harmonic analysis and coding theory to derive bounds on code size. We present a general framework using group representation theory and Bochner theorem, which transforms coding bounds into semidefinite programs. This approach recovers the classical Delsarte LP bound for symmetric spaces, recover traditional results like MacWilliam's identities. The analysis utilizes the Bochner's theorem, which is essentially a "Fourier transform" on another function space, and this provides a useful SDP-based framework with variant applications.

Keywords: Semidefinite programming, Bochner theorem, Hamming space, Kissing number

1 Introduction and Problem Statement

In coding theory, a fundamental extremal problem is determining the maximum cardinality $A(X, \delta)$ of a code C in a metric space X with minimum distance at least δ:

$$A(X, \delta) := \max\{|C| : d(c, c') \geq \delta \text{ for all distinct } c, c' \in C\}.$$

The classical **linear programming method**, introduced by Delsarte for association schemes and extended to two-point homogeneous spaces, provides powerful upper bounds. For such a space, there exists a family of orthogonal polynomials $\{P_k\}$ such that for any code C,

$$\sum_{(c,c')\in C^2} P_k(d(c, c')) \geq 0, \quad k = 0, 1, 2, \ldots$$

These inequalities, expressed via the distance distribution of C, form the linear constraints of the Delsarte LP.

However, many spaces of modern interest in coding theory (e.g., Grassmann manifolds, ordered Hamming spaces, non-binary Johnson spaces) are not two-point homogeneous. Furthermore, even for classical spaces like the binary Hamming space, the LP bound can be improved by exploiting constraints involving triples of points rather than just pairs.

So, **how can we systematically construct upper bounds for $A(X, \delta)$ for a wide class of spaces, leveraging their symmetry?** The answer is using harmonic analysis on spaces X under the action of a compact group G, and the representation theory of $\mathcal{C}(X)$.

1.1 Technical Overview

A positive semi-definite continuous function on a compact space X is a function $F \in \mathcal{C}(X \times X)$ such that $F(x, y) = \overline{F(y, x)}$ and for all n and $(x_1, \ldots, x_n) \in X^n$, for all $(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$,

$$\sum_{i,j\in[n]} \alpha_i F(x_i, x_j)\overline{\alpha_j} \geq 0.$$

This property will be denoted $F \succeq 0$.

We focus on the upper bound for $A(X, \delta)$, and it follows the program:

$$\begin{aligned} m(X, \delta) := \inf\{t : \quad & F \in \mathcal{C}(X^2), \overline{F} = F, F \succeq 0 \\ & F(x, x) \leq t - 1, \text{ forall } x \\ & F(x, y) \leq -1, \text{ forall } x, y \text{ satisfying } d(x, y) \geq \delta\}. \end{aligned} \tag{1}$$

For any feasible F, if the minimum distance of any two elements in C is greater then δ, then we have

$$0 \leq \sum_{(c,c')\in C\times C} F(c, c') \leq |C|(t - 1 - (|C| - 1)),$$

due to the positive semi-definite property of F. Thus $t \geq |C|$. Further if F is the optimal choice, we have $m(X,\delta) \geq |C|$. Taking the maximum on the right side yields $m(X,\delta) \geq A(X,\delta)$.

Further, we consider G acting on X. From a feasible F, we can construct a G-invariant feasible $F^*(x,y) = \int_G F(gx,gy)\mathrm{d}g$, which satisfies all the conditions and yields the same solution t.

To sum up the above, the problem boils down to establishing the structure of $F \in \mathcal{C}(X^2)$ where $\overline{F} = F, F \succeq 0$ and is G-invariant, for specific spaces X.

2 Harmonic Analysis Framework

We omit the discussion of linear representation of finite groups and turn to generalize it to the compact groups.

A compact group G affords a Haar measure ν, which is left and right invariant. A finite-dimensional representation of a compact group G is a **continuous** homomorphism $\rho : G \mapsto \mathrm{GL}(V)$ where V is a finite-dimension conplex vector space. Every inner product $\langle\cdot,\cdot\rangle$ is averaged to obtain a G-invariant one:

$$\langle x,y\rangle_H = \int_G \langle gx,gy\rangle \nu(\mathrm{d}g).$$

So the Maschke theorem still holds for compact groups, but there are infinitely many irreducible representations, we denote the set of irreducible representations as $\mathcal{R} = \{R_i\}_{i\geq 0}$.

We have the orthogonality for entries of matrices of the irreducible representations of G where $R_{i,j} = \langle R_g e_j, e_i\rangle_H$:

$$\langle R_{ij}, S_{kl}\rangle = \frac{1}{d_R}\delta_{ik}\delta_{jl}.$$

The Peter-Weyl Theorem asserts that these elements span a vector space which is dense in $\mathcal{C}(G)$ in the topology of uniform convergence. The representation is denoted $(L, \mathcal{C}(G))$ where

$$L_g f(x) = f(g^{-1}x), \quad R_g f(x) = f(xg)$$

define the operators L_g, R_g, namely, the left and right regular representations. They are unitary for all $g \in G$ because

$$\langle L_g f, L_g f'\rangle_G = \int_G f(g^{-1}x)\overline{f'(g^{-1}x)}\nu(\mathrm{d}x) = \int_G f(y)\overline{f'(y)}\nu(\mathrm{d}y) = \langle f, f'\rangle_G,$$

and similar for R_g.

2.1 Generalized Peter-Weyl Theorem and the Structure of $\mathcal{C}(X)$

The Peter-Weyl theorem, which establishes the orthogonal decomposition of the regular representation $\mathcal{C}(G)$ of a compact group G into finite-dimensional irreducible components, generalizes naturally to function spaces on homogeneous spaces. This generalization forms the cornerstone of harmonic analysis on symmetric spaces and provides the fundamental framework for applying representation theory to coding theory problems on spaces such as spheres and Grassmannians.

Let G be a compact group acting continuously and transitively on a compact space X; such a space is called a G-**homogeneous space**. Fix a base point $x_0 \in X$ and let $H = \{g : gx_0 = x_0\}$ denote its stabilizer subgroup. Then X can be identified with the quotient space G/H via the map $\tau : gH \mapsto g \cdot x_0$. The normalized Haar measure ν on G induces a G-invariant measure μ on X, defined by $\mu(A) = \nu(\pi^{-1} \circ \tau^{-1}(A))$ where $\pi : g \mapsto gH$ be the quotient mapping of G with respect to subgroup H. We have $\mu(X) = 1$ and μ is G-invariant. Thus, the space $\mathcal{C}(X)$ of complex-valued continuous functions on X is endowed with the G-action $(g \cdot f)(x) = f(g^{-1}x)$ and the G-invariant inner product

$$\langle f, f'\rangle_X := \int_X f(x)\overline{f'(x)}\,\mu(\mathrm{d}x).$$

Let $\mathcal{R} = \{R_k\}_{k\geq 0}$ be a complete set of pairwise non-isomorphic finite-dimensional irreducible unitary representations of G, with R_0 denoting the trivial representation and $d_k = \dim R_k$. By Peter-Weyl, we have the orthogonal decomposition for $\mathcal{C}(G)$

$$\mathcal{C}(G) = \bigoplus_{k\geq 0} \mathcal{I}_k, \quad \mathcal{I}_k \simeq R_k^{d_k},$$

where $\mathcal{I}_k$ is the isotypic component associated to R_k.

The key observation is that $\mathcal{C}(X)$ can be identified with the subspace of $\mathcal{C}(G)$ consisting of functions that are right-H-invariant:

$$\mathcal{C}(X) \simeq \{f \in \mathcal{C}(G) : f(gh) = f(g) \text{ for all } h \in H, g \in G\}.$$

Every component $H_{k,i}$ of $\mathcal{I}_k$ are all L-invariant subspaces. We have for all $g \in G$, $L_g H_{k,i} = H_{k,i}$. Since L_g and R_h commute for all $h, g \in G$, $R_h H_{k,i}$ is also an L-invariant subspace. If $R_h H_{k,i} = H_{k,i}$ for all $h \in G$, then it is among the factorization of $\mathcal{C}(X)$. Consequently, $\mathcal{C}(X)$ decomposes orthogonally as

$$\mathcal{C}(X) = \bigoplus_{k \geq 0} \mathcal{I}_k^X, \quad \text{where } \mathcal{I}_k^X \simeq R_k^{m_k}.$$

Here $0 \leq m_k \leq d_k$. Each isotypic component $\mathcal{I}_k^X$ can be further decomposed into m_k pairwise orthogonal G-submodules, each isomorphic to R_k:

$$\mathcal{I}_k^X = H_{k,1} \perp H_{k,2} \perp \cdots \perp H_{k,m_k}, \quad H_{k,i} \simeq R_k.$$

This decomposition holds in the sense of both uniform convergence and L^2 convergence. In particular, if we choose for each k an orthonormal basis $\{e_{k,i,s} : 1 \leq i \leq m_k,\ 1 \leq s \leq d_k\}$ of $\mathcal{I}_k^X$ such that the action of G on $H_{k,i}$ is given by the matrices of R_k in the basis $\{e_{k,i,s}\}_{s=1}^{d_k}$, then the set $\{e_{k,i,s}\}$ forms a complete orthonormal system for $L^2(X, \mu)$.

Theorem 2.1. *Let $\pi : G \to U(H)$ be a continuous homomorphism from a compact group G to the unitary group of a Hilbert space H. Then H is a direct Hilbert sum of finite-dimensional irreducible G-modules.*

Proof. We construct a non-zero finite-dimensional G-invariant subspace of H and then iterate with its orthogonal complement.

Fix a non-zero vector $v \in H$. Choose a continuous function $f \in \mathcal{C}(G)$ such that

$$\int_G f(g)(\pi(g)v)\, dg \neq 0.$$

Such an f exists because the constant function $f \equiv 1$ already gives $\int_G \pi(g)v\, dg$, which is non-zero if $v \neq 0$ (by averaging properties of the Haar measure on compact groups).

By the Peter-Weyl theorem, the matrix coefficients $R_{i,j}$ of finite-dimensional irreducible representations are dense in $\mathcal{C}(G)$. Hence we may assume that f is a finite linear combination of such matrix coefficients:

$$f(g) = \sum_{k,i,j} c_{k,i,j} R_{i,j}^{(k)}(g) = \sum_{k,i,j} c_{k,i,j} \langle e_i, R^{(k)}(g) e_j \rangle,$$

where each $R^{(k)}$ is a finite-dimensional unitary irreducible representation of G.

Thus there exists a finite-dimensional unitary representation (ρ, V) of G and vectors $u, w \in V$ such that

$$f(g) = \langle \rho(g^{-1})u, w \rangle_V.$$

Here (ρ, V) is the direct sum of many representations, where the inner product is defined the sum of the inner products of all the spaces.

Now define an operator $T : V \to H$ by

$$T(x) = \int_G \langle \rho(g^{-1})x, w \rangle_V\, (\pi(g)v)\, dg.$$

One checks that T commutes with the action of G: for any $h \in G$,

$$T(\rho(h)x) = \int_G \langle \rho(g^{-1}h)x, w \rangle_V\, (\pi(g)v)\, dg = \int_G \langle \rho(g^{-1})x, w \rangle_V\, (\pi(h)\pi(g)v)\, dg = \pi(h)T(x).$$

Moreover, T is non-zero because

$$T(u) = \int_G \langle \rho(g^{-1})u, w \rangle_V\, (\pi(g)v)\, dg = \int_G f(g)(\pi(g)v)\, dg \neq 0.$$

Therefore the image $\mathrm{Im}(T)$ is a non-zero finite-dimensional G-invariant subspace of H. $\square$

2.2 Commuting Endomorphism and Zonal Matrices $E_k(x,y)$

In this section, we utilize the representation tools to study the algebra

$$\mathrm{End}_G(\mathcal{C}(X)) := \{T : \mathcal{C}(X) \mapsto \mathcal{C}(X) \mid T(L_g f) = L_g(Tf), \forall g \in G, f \in \mathcal{C}(X)\}.$$

A continuous function $K \in \mathcal{C}(X^2)$ is called ***G*-invariant** or **zonal** if $K(gx, gy) = K(x, y)$ for all $g \in G$, and let

$$\mathcal{K} = \{K \in \mathcal{C}(X^2) : K(gx, gy) = K(x, y),\ \forall g \in G,\ \forall (x, y) \in X \times X\}.$$

The triple $(\mathcal{K}, +, *)$ is a $\mathbb{C}^*$-algebra, with $*$ defined to be the convolution

$$(K * K')(x, y) = \int_X K(x, z)K'(z, y)\mu(\mathrm{d}z).$$

To such a kernel, we associate the **Hilbert-Schmidt operator** T_K defined by

$$(T_K f)(x) = \frac{1}{\mu(X)} \int_X K(x, y)f(y)\, d\mu(y).$$

If K is zonal, then $T_K \in \mathrm{End}_G(\mathcal{C}(X))$. So there is an embedding $K \mapsto T_K$ from $\mathcal{K}$ to $\mathcal{C}(X)$.

For each isotypic component $V = \mathcal{I}_k^X \simeq \bigoplus_{i=1}^{m_k} W_{k,i}$ of $\mathcal{C}(X)$, $W_{k,i} \simeq R_k$, we have an orthonormal basis $(e_{k,i,1}, \ldots, e_{k,i,d_k})$ of $W_{k,i}$, and the operator L_g is expressed by a unitary matrix $R(g)$. We define an $m_k \times m_k$ matrix $E_k(x, y)$ with entries

$$E_{k,i,j}(x, y) := \sum_{s=1}^{d_k} e_{k,i,s}(x)\, \overline{e_{k,j,s}(y)}.$$

By the definition, we know that $E_k(x, y)$ is stable when the orthonormal basis of $W_{k,i}$ changes. With some calculation, we reach the results as follows.

Proposition 2.2. *The functions $E_{k,i,j}$ satisfy the following properties:*

(a) $E_{k,i,j}$ is zonal: $E_{k,i,j}(gx, gy) = E_{k,i,j}(x, y)$.

(b) $T_{k,i,j} := T_{E_{k,i,j}}$ then $T_{k,i,j}(W_{k,l}) = \delta_{jl} W_{k,i}$.

(c) $T_{k,i,j} \circ T_{k,j,l} := T_{E_{k,i,j} * E_{k,j,l}} = T_{k,i,l}$.

Theorem 2.3. *For $V \subset \mathcal{C}(X)$ be a sub-G-module, if $V \simeq R^m$, then*

$$\mathcal{K}_V \simeq \mathrm{End}_G(V) \simeq \mathbb{C}^{m\times m},$$

RC where

$$\mathcal{K}_V := \{K(x, y) = f(x)\overline{g(y)} : f, g \in V\} \cap \mathcal{K}$$

The isomorphism $\mathbb{C}^{m_k\times m_k} \mapsto \mathrm{End}_G(V)$ is defined by:

$$A \mapsto \sum_{i,j\in[m_k]} a_{j,i} T_{k,j,i} p_{W_{k,i}},$$

where $p_W f$ is the projection of f onto the subspace $W \subset V$. This isomorphism is also written in the form

$$A \mapsto T_{\langle A, \overline{E}_k\rangle},$$

noticing that by Proposition 2.2(b), $T_{k,j,i}(f) = T_{k,j,i}(p_{W_{k,i}} f)$. Here is the proof that the mapping is onto. Take any $\phi \in \mathrm{End}_G(V)$, then $\phi(W_{k,i})$ is a G-invariant subspace, and $p_{W_{k,j}} \phi|_{W_{k,i}}$ is a G-invariant linear mapping from $W_{k,i}$ to $W_{k,j}$. By Schur's lemma, the mapping is some multiple of $T_{k,j,i}$. By the orthogonality, we have $\phi = \sum_{i,j\in[m_k]} a_{j,i} T_{k,j,i}$ for some $(a_{i,j})_{i,j\in[m_k]}$.

The isomorphism $\mathcal{K}_V \mapsto \mathrm{End}_G(V)$ is defined by $K \mapsto T_K$. Since $A \mapsto T_{\langle A, \overline{E}_k\rangle}$ is an isomorphism, every T is some $T_{\langle A, \overline{E}_k\rangle}$. Thus $K = \langle A, \overline{E}_k\rangle$ is a combination of $f(x)\overline{g(y)}$, thus in $\mathcal{K}_V$.

Moreover, $\mathrm{End}_G(\mathcal{C}(X))$ is commutative if and only if the decomposition is **multiplicity-free** ($m_k \leq 1$ for all k), which happens for ***G*-symmetric spaces**.

2.3 Bochner Theorem for G-Invariant Positive Semi-Definite Functions

Definition 2.4 (Positive Semi-Definite Functions)**.** A continuous function $F : X \times X \to \mathbb{C}$ is **positive semi-definite** if $F(x,y) = \overline{F(y,x)}$, and for any finite set $\{x_1, \dots, x_n\} \subset X$, the matrix $\big(F(x_i, x_j)\big)$ is positive semidefinite.

Or equivalently, for all $\alpha \in \mathcal{C}(X)$,

$$\int_{X \times X} \alpha(x) F(x,y) \overline{\alpha(y)} \mu(\mathrm{d}x, \mathrm{d}y) \geq 0.$$

Theorem 2.5. *(Bochner) $F \in \mathcal{C}(X^2)$ is a G-invariant continuous positive semi-definite function if and only if it admits the expansion*

$$F(x,y) = \sum_{k \geq 0} \langle F_k, \overline{E_k(x,y)} \rangle,$$

where each F_k is an $m_k \times m_k$ Hermitian positive semidefinite matrix

$$F_k = \frac{1}{d_k \mu(X^2)} \int_{X^2} F(x,y) \overline{E_k(x,y)} \mu(\mathrm{d}x, \mathrm{d}y) \succeq 0.$$

*The sum converges to F in the L^2 topology; if G acts **homogeneously** on X, the sum converges uniformly.*

Proof. The elements $e_{k,i,s}(x) e_{l,j,t}(y)$ form a complete system of orthonormal elements of $\mathcal{C}(X^2)$. Hence F has a decomposition

$$F(x,y) = \sum_{k,i,s,l,j,t} f_{k,i,s,l,j,t} \, e_{k,i,s}(x) \overline{e_{l,j,t}(y)}$$

where the convergence of the sum is L^2. The condition $F(gx, gy) = F(x,y)$ translates to:

$$f_{k,i,u,l,j,v} = \sum_{s,t} f_{k,i,s,l,j,t} R_{k,u,s}(g) \overline{R_{l,v,t}(g)}.$$

Integrating on $g \in G$ and applying the orthogonality relations of shows that $f_{k,i,u,l,j,v} = 0$ if $k \neq l$ or $u \neq v$. Moreover it shows that $f_{k,i,u,k,j,u}$ does not depend on u. The resulting expression of F reads:

$$F(x,y) = \sum_{k \geq 0} \left(\sum_{i,j} f_{k,i,j} E_{k,i,j}(x,y) \right)$$

and

$$d_k f_{k,i,j} = \frac{1}{\mu(X^2)} \int_{X^2} F(x,y) \overline{E_{k,i,j}(x,y)} \, d\mu(x) d\mu(y),$$

which is the wanted expression, with $F_k := (f_{k,i,j})_{1 \leq i,j \leq m_k}$.

Now we show that $F_k \succeq 0$. Let, for k, s fixed, $\alpha(x) = \sum_i \alpha_i e_{k,i,s}(x)$, with $\sum_i |\alpha_i|^2 < +\infty$. By Definition 2.4 for $\alpha \in \mathcal{C}(X)$, We compute:

$$\begin{aligned}
\int_{X^2} \alpha(x) F(x,y) \overline{\alpha(y)} \, \mu(\mathrm{d}x, \mathrm{d}y) &= \sum_{k \geq 0} \sum_{i,j \in [m_k]} \int_{X^2} F_{k,i,j} E_{k,i,j}(x,y) \alpha(x) \overline{\alpha(y)} \, \mu(\mathrm{d}x, \mathrm{d}y) \\
&= \sum_{k \geq 0} \sum_{i,j \in [m_k]} \sum_{s \in [d_k]} \int_{X^2} F_{k,i,j} e_{k,i,s}(x) \alpha(x) \overline{e_{k,j,s}(y) \alpha(y)} \, \mu(\mathrm{d}x, \mathrm{d}y) \\
&= \sum_{k \geq 0} \sum_{i,j \in [m_k]} \sum_{s \in [d_k]} F_{k,i,j} \alpha_{i,s} \overline{\alpha_{j,s}} \geq 0,
\end{aligned}$$

thus $F_k \succeq 0$. Thus, the "if" direction is proved. □

Remark 2.6. This theorem translates the infinite-dimensional condition $F \geq 0$ into a family of finite-dimensional matrix constraints $F_k \succeq 0$. ◊

3 SDP Upper Bound for Codes

Using the Bochner theorem, we can transform the abstract upper bound problem into an SDP.

First, we restate Eq. (1) with additional restrictions mentioned in Section 1.1:

$$\begin{aligned} m(X,\delta) := \inf\{t:\ & F \in \mathcal{C}(X^2),\ \overline{F} = F,\ F \succeq 0,\ F \text{ is } G\text{-invariant}, \\ & F(x,x) \le t-1,\ \text{forall } x \\ & F(x,y) \le -1,\ \text{forall } x,y \text{ satisfying } d(x,y) \ge \delta\}. \end{aligned}$$

The Bochner theorem is an infinite sum, but we need a computationally tractable finite-dimensional SDP. The central idea is to exploit the density of a well-chosen sequence of finite-dimensional G-subspaces within $\mathcal{C}(X)$. Assume we have an increasing sequence $(V_d)_{d\ge 0}$ of such subspaces, with $\bigcup_{d\ge 0} V_d$ being dense in $\mathcal{C}(X)$ under uniform convergence. The constructive strategy is then to project the G-irreducible decomposition of $\mathcal{C}(X)$ onto these finite-dimensional pieces. For a given degree d, we consider the isotypic decomposition of V_d:

$$V_d = \bigoplus_{k\ge 0} \mathcal{I}_k^{(d)}, \quad \text{where } \mathcal{I}_k^{(d)} \simeq R_k^{m_{d,k}} \text{ and } m_{d,k} \le m_k.$$

For each such truncated isotypic component $\mathcal{I}_k^{(d)}$, we construct its associated zonal matrix $E_k^{(d)}(x,y)$ of size $m_{d,k} \times m_{d,k}$ analogously to the infinite-dimensional case. We have the following theorem.

Theorem 3.1. *Let X be a compact space, μ a finite G-invariant Borel measure on X, and G a compact group acting continuously on X. Assume there exists an increasing sequence $(V_d)_{d\ge 0}$ of finite-dimensional G-subspaces of $\mathcal{C}(X)$ whose union $\bigcup_{d\ge 0} V_d$ is dense in $\mathcal{C}(X)$ for the uniform topology. Suppose moreover that X is homogeneous under a compact group Γ containing G (so $X = \Gamma/\Gamma_0$ where for given x_0, $\Gamma_0 = \{g : gx_0 = x_0\}$).*

For each d, let $\mathcal{I}_k^{(d)}$ be the isotypic component of V_d corresponding to the irreducible representation R_k of G, with multiplicity $m_{d,k}$ ($0 \le m_{d,k} \le m_k$, where m_k is the multiplicity of R_k in $\mathcal{C}(X)$). Let $E_k^{(d)}(x,y)$ be the $m_{d,k} \times m_{d,k}$ zonal matrix attached to $\mathcal{I}_k^{(d)}$, constructed from an orthonormal basis of each irreducible copy inside $\mathcal{I}_k^{(d)}$ in which G acts by the matrices of R_k.

Then every continuous G-invariant positive semi-definite function $F \in \mathcal{C}(X \times X)$ is the uniform limit of a sequence $(F_d)_{d\ge 0}$ of G-invariant positive semi-definite functions of the form

$$F_d(x,y) = \sum_{k\ge 0} \langle F_{d,k}, \overline{E_k^{(d)}(x,y)} \rangle,$$

where each $F_{d,k}$ is a positive semidefinite matrix of size $m_{d,k} \times m_{d,k}$, and the sum has only finitely many non-zero terms for each fixed d. The inner product $\langle \cdot,\cdot \rangle$ denotes the Frobenius inner product of matrices.

Applying Theorem 3.1 and assuming we have a nested sequence of finite-dimensional G-subspaces $V_d \subset \mathcal{C}(X)$ whose union is dense, we obtain a sequence of finite-dimensional SDPs:

$$\begin{aligned} m^{(d)}(X,\delta) := \inf\{t:\ & \forall k \ge 0, F_k^{(d)} \succeq 0, \\ & \sum_{k\ge 0} \langle F_k^{(d)}, \overline{E_k^{(d)}(x,x)} \rangle \le t-1,\ \text{forall } x \\ & \sum_{k\ge 0} \langle F_k^{(d)}, \overline{E_k^{(d)}(x,y)} \rangle \le -1 \text{ forall } x,y \text{ satisfying } d(x,y) \ge \delta\}. \end{aligned}$$

where the $F_k^{(d)} \in \mathbb{C}^{m_{d,k}\times m_{d,k}}$'s are the corresponding coefficient of the function $T_L F$ in the proof of Theorem 3.1. We have $m(X,\delta) \le m^{(d)}(X,\delta)$ and $\lim_{d\to\infty} m^{(d)}(X,\delta) = m(X,\delta)$.

Since $E_k(x,y)$'s are G-invariant, if (x,y) and (x',y') belongs to the same orbit of G acting on X^2, they have the same $E_k(x,y) = E_k(x',y')$. Thus, if a set of parameters $u = (u_i)$ is used to determine different orbits, we have $E_k(x,y) = Y_k(u(x,y))$ for some function Y_k.

Here is an application of this SDP framework, which relates graphs to codes.

3.1 From Codes to Graphs and the ϑ Number

This section establishes a fundamental link between coding–theoretic extremal problems and a celebrated invariant from graph theory: the **Lovász ϑ number**. The central idea is to view a code with a prescribed minimum distance as an independent set in a certain distance graph, thereby allowing graph–theoretic methods to bound the code's size.

Let X be a finite metric space with distance d. For a given $\delta > 0$, define the graph $\Gamma(X, \delta)$ with vertex set X and an edge between x and y if $0 < d(x, y) < \delta$. A code $C \subset X$ with minimum distance at least δ is precisely an independent set in this graph. Thus, the maximum code size $A(X, \delta)$ equals the independence number $\alpha(\Gamma(X, \delta))$, which is NP–hard to compute.

Consider a graph $\Gamma = (V = [n], E)$, and S be an independent set. Let

$$M_S(x, y) = \frac{1}{|S|}\mathbf{1}_S(x)\mathbf{1}_S(y),$$

we know that if $(x, y) \in E$ then $M(x, y) = 0$. Further, $\sum_{x\in V} M_S(x, x) = 1$ and $M_S \succeq 0$ since it is a rank-1 matrix.

This gives us the intuition of Lovász ϑ number, defined via SDP.

Definition 3.2. The theta number of the graph $\Gamma = (V, E)$ with $V = [n]$ is defined

$$\begin{aligned} \vartheta(\Gamma) = \max_{B_{ij}} \quad & \sum_{i,j\in[n]} B_{ij} \\ \text{subject to} \quad & B \in \mathbb{R}^{n\times n},\ B \succeq 0, \\ & \sum_i B_{ii} = 1, \\ & B_{ij} = 0, \quad \forall (i, j) \in E. \end{aligned}$$

The dual program is

$$\begin{aligned} \min_{B_{ij}} \quad & t \\ \text{subject to} \quad & B \in \mathbb{R}^{n\times n},\ B \succeq 0, \\ & B_{ii} = t - 1, \quad \forall i \in V. \\ & B_{ij} = -1, \quad \forall (i, j) \notin E. \end{aligned}$$

Theorem 3.3.

$$\alpha(\Gamma) \le \vartheta(\Gamma) \le \chi(\overline{\Gamma}).$$

Proof. The first inequality is because M_S for an independent set is a feasible matrix.

Let $\varphi : V \mapsto [k]$ be a coloring of $\overline{\Gamma}$, thus C is a feasible matrix of the **dual program** where $C_{ii} = k - 1$ for all $i \in V$, $C_{ij} = -1$ if $\varphi(i) \ne \varphi(j)$, and $C_{ij} = 0$ otherwise. Thus $k \ge \vartheta(\Gamma)$. □

These SDPs connect well to those in coding theory.

4 Recovering the LP Bound: 2-Point Homogeneous Spaces

In this section, we show that the SDP in context of 2-homogeneous spaces is reduced to LP bounds. Further, we recover the Delsarte LP bound regarding the linear code in $\mathbb{F}_q^n$.

Definition 4.1 (2-Point Homogeneous Spaces)**.** A metric space (X, d) is said to be **2-point homogeneous** under the action of a group G if G acts transitively on X, leaves the distance d invariant, and if for any two pairs (x, y) and (x', y') in X^2,

$$\exists g \text{ s.t. } (gx, gy) = (x', y') \iff d(x, y) = d(x', y').$$

Prominent examples of such spaces in coding theory include the Hamming space H_n and the Johnson space J_n^w under the Hamming distance, as well as the unit sphere S^{n-1} under the angular distance, with their respective symmetry groups.

The definition means that the orbits of G acting on X^2 are completely determined by the distance between the point pair. As a result, the associated zonal matrix $E_k(x,y)$ reduces to a single coefficient, which is a function of the distance $d(x,y)$, denoted $P_k(t)$ with $t = d(x,y)$.

Here, we give another class of space, which we will discuss later:

Definition 4.2 (G-Symmetric Spaces)**.** A space X is G-symmetric if for all $(x,y) \in X^2$, there exists $g \in G$ such that $gx = y, gy = x$, i.e. (x,y) and (y,x) belongs to the same orbit og G acting on X^2.

Proposition 4.3. *A space X has $\mathcal{C}(X) = \bigoplus_{k\geq 0} R_k^{m_k}$ where each irreducible representation R_k has dimension d_k. If X is G-symmetric, $m_k = 0$ or 1 for all $k \geq 0$, and $E_k(x,y)$ is real symmetric.*

Proof. Recall that $\mathcal{K}$ is the algebra of the G-invariant functions, with convolution $*$. Now X is G-invariant, thus $K(x,y) = K(y,x)$, and

$$(K' * K)(x,y) = \int_X K'(x,z)K(z,y)\mu(\mathrm{d}z) = (K * K')(y,x) = (K * K')(x,y).$$

By Theorem 2.3, for $V_k = R_k^{m_k}$, $\mathcal{K}_{V_k} \simeq \mathrm{End}_G(V_k) \simeq \mathbb{C}^{m_k \times m_k}$. Thus $m_k = 0$ or 1.

Besides, $E_k(x,y) = E_k(y,x) = \overline{E_k(x,y)}$, thus real symmetric. □

The 2-point homogeneous spaces are naturally G-symmetric spaces, due to the symmetric property of the metric $d(\cdot,\cdot)$.

From Proposition 4.3, for a 2-point homogeneous space X, the decomposition of the space of continuous functions $\mathcal{C}(X)$ under the action of G is multiplicity-free

$$\mathcal{C}(X) = \bigoplus_{k\geq 0} H_k.$$

This function, denoted $P_k(t)$ with $t = d(x,y)$, is called the **zonal function** associated to the irreducible subspace H_k. The sequence $\{P_k\}$ forms a family of orthogonal polynomials with respect to the measure induced on the distance variable by the G-invariant measure on X: since H_k's are mutually orthogonal, the basis vectors are also mutually orthogonal. Thus,

$$\langle e_{k,1,s}, e_{l,1,t} \rangle = \delta_{kl}\delta_{st},$$

and as a result,

$$\int_{X^2} E_k(x,y)\overline{E_l(x,y)}\mu(\mathrm{d}x,\mathrm{d}y) = \delta_{kl}$$

so $P_k(t)$'s are also mutually orthogonal:

$$\int_D P_k(t)\overline{P_l(t)}w(t)\mathrm{d}t = \delta_{kl}.$$

where $D = \{d(x,y) : x,y \in X\}$ and $w(t)\mathrm{d}t = \int_{d(x,y)\in[t,t+\mathrm{d}t)} \mu(\mathrm{d}x,\mathrm{d}y)$ is the weight of the orbit under measure μ.

A G-invariant positive definite function on X^2 can be expressed as a nonnegative linear combination

$$F(x,y) = \sum_{k\geq 0} f_k P_k(d(x,y)), \quad f_k \geq 0.$$

This characterization leads to the classical Delsarte LP bound for the maximum size $A(X,\delta)$ of a code with minimum distance at least δ. The orthogonality of the polynomials P_k is key to their determination and to the computational aspects of the bound. Examples include the Krawtchouk polynomials for the Hamming space, the Hahn polynomials for the Johnson space, and the Gegenbauer polynomials for the unit sphere. These one-variable orthogonal polynomials allow the infinite-dimensional cone of positive definite functions to be described by a simple sequence of nonnegative coefficients, making the LP method particularly effective for 2-point homogeneous spaces.

Since $m_k = 1$ and $E_k(x,y) = P_k(d(x,y))$, the SDP reduces exactly to the Delsarte LP, where $e = d(x,x)$ for all x:

$$\begin{aligned} m(X,\delta) = \min_{(f_t)_{t\in\mathbb{N}}} \quad & 1 + \sum_{k\geq 0} f_k P_k(e) \\ \text{subject to} \quad & \sum_{k\geq 0} f_k P_k(t) \leq -1, \quad t \in D_{\geq\delta}, \\ & f_t \geq 0, \quad t \in \mathbb{N}. \end{aligned} \tag{2}$$

where $D_{\geq\delta} = \{d(x,y) : x, y \in X, d(x,y) \geq \delta\}$.
Additionally, we have since $E_k(x,x) = E_k(gx, gx)$ for all $g \in G$,

$$P_k(e) = E_k(x,x) = \int_X E_k(x,x)\mu(\mathrm{d}x) = \sum_{s=1}^{d_k} \int_X e_{k,1,s}(x)\overline{e_{k,1,s}(x)}\mu(\mathrm{d}x) = d_k.$$

Then, with Eq. (1), if $f_k \geq 0$ and $1 + \sum_{k\geq 0} f_k P_k(t) \leq 0$ for all $t \in D_{\geq\delta}$,

$$A(X,\delta) \leq m(X,\delta) \leq 1 + \sum_{k\geq 0} f_k P_k(e) = 1 + \sum_{k\geq 0} f_k d_k.$$

Theorem 4.4. *Let $F(t) = b + \sum_{k=0}^{d} f_k P_k(t)$. If $f_k \geq 0$ for all $0 \leq k \leq d$ and $b > 0$, and if $F(t) \leq 0$ for all $t \in D_{\geq\delta}$, then*

$$A(X,\delta) \leq \frac{F(e)}{b}.$$

Proof. From the premise we know that $\left(\frac{f_i}{b}\right)_{i\geq 0}$, where $f_k = 0$ for all $k > d$, is a feasible solution for the linear program Eq. (2). Thus $A(X,\delta) \leq m(X,\delta) \leq 1 + \sum_{k=0}^{d} \frac{f_k}{b} P_k(e) = \frac{F(e)}{b}$. □

The following subsections select $\mathbb{F}_q^n$ and n-dimensional sphere S^{n-1} as two examples, and recover some relative results.

4.1 Connection to Original Delsarte Bound: q-Hamming Space

In this subsection, we consider $G = S_q^n \rtimes S_n$ acting on $X = H_{n,q} = \mathbb{F}_q^n = \{0, \ldots, q-1\}^n$. We will develop a theory of codes with Hamming distance on $\mathbb{F}_q$, establish the polynomial $P_k(t)$, and develop the MacWilliams identities. Furthermore, we use MacWilliams identities to develop a traditional LP Bound, and relate this to our SDP scheme.

Recall that S_q acts on $\mathbb{F}_q$ naturally, and S_n acts on $H_{n,q}$ by $\sigma(a_1, \ldots, a_n) = (a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n)})$.

4.1.1 Harmonic analysis on $H_{n,q}$

Let $\mathbb{F}_q$ be the alphabet. The action of S_q on $\mathcal{C}(F)$ gives the decomposition

$$\mathcal{C}(F) = \mathbb{C}\mathbf{1} \perp L$$

where L is irreducible of dimension $q-1$. Choose an orthonormal basis $z_0 = \mathbf{1}, z_1, \ldots, z_{q-1}$ of $\mathcal{C}(F)$ with z_0 invariant and $z_1, \ldots, z_{q-1}$ spanning L.

For $x = (x_1, \ldots, x_n) \in H_{n,q}$ and $\phi = (\phi_1, \ldots, \phi_n)$ with each $\phi_i \in \{z_0, \ldots, z_{q-1}\}$, define

$$\phi(x) = \prod_{i=1}^{n} \phi_i(x_i).$$

The weight of ϕ is defined $\mathrm{wt}(\phi) = |\{i : \phi_i \neq z_0\}|$. Under the inner product $\langle \phi, \phi' \rangle = \prod_{i=1}^{n} \langle \phi_i, \phi_i' \rangle$, the set

$$S = \left\{ \phi : \phi(x) = \prod_{i=1}^{n} \phi_i(x_i), \phi_i \in \{z_0, \ldots, z_{q-1}\} \right\}$$

is a orthonormal basis of $\mathcal{C}(H_{n,q})$.

For $0 \le k \le n$, let P_k be the subspace spanned by all ϕ with $\mathrm{wt}(\phi) = k$. Then

$$\mathcal{C}(H_{n,q}) = P_0 \perp P_1 \perp \cdots \perp P_n,$$

and each P_k is G-irreducible of dimension $(q-1)^k\binom{n}{k}$.

The G-orbits on X^2 are parametrized by the Hamming distance $t = d_H(x,y) \in \{0,1,\ldots,n\}$. Hence the zonal function associated to P_k depends only on t:

$$E_k(x,y) = \sum_{\phi,\, \mathrm{wt}(\phi)=k} \phi(x)\overline{\phi(y)} =: P_k(t), \quad t = d_H(x,y).$$

With the normalization $z_0(0) = 1$, one computes (the details are omitted)

$$P_k(t) = \sum_{i=0}^{k} \binom{t}{i}\binom{n-t}{k-i}(-1)^i(q-1)^{k-i}.$$

These are the **Krawtchouk polynomials** $K_k^{n,q}(t)$ for the q-ary Hamming scheme. They satisfy:

1. $\deg(K_k^{n,q}) = k$,
2. $K_k^{n,q}(0) = (q-1)^k\binom{n}{k}$,
3. Orthogonality:
$$\frac{1}{q^n}\sum_{w=0}^{n}\binom{n}{w}K_k^{n,q}(w)K_l^{n,q}(w) = \delta_{k,l}\binom{n}{k}(q-1)^k.$$

Thus the G-invariant positive definite functions on $H_{n,q}$ are exactly

$$F(x,y) = \sum_{k=0}^{n} f_k K_k^{n,q}(d_H(x,y)), \quad f_k \ge 0.$$

4.1.2 MacWilliams Identities and the Delsarte LP Bound

Let $C \subset H_{n,q}$ be a linear code. The dual code is defined

$$C^\perp = \left\{ y \in H_{q,n} : \sum_{i=1}^{n} x_i y_i = 0 \right\}.$$

Define the **distance distribution** of code C

$$A_w(C) = \frac{1}{|C|}|\{(x,x') \in C^2 : d_H(x,x') = w\}|,$$

which can be regarded as the number of the w-weight codes. Here we use $\phi_z(x) = \omega_q^{x\cdot z}$, and we have

$$\langle \mathbf{1}_C, \phi_z\rangle = \frac{1}{q^n}\sum_{x\in C}\overline{\phi_z(x)} = \frac{|C|}{q^n}\mathbf{1}_{C^\perp}(z).$$

Thus, we compute the distance distribution of $C^\perp$. Notice that in a linear code C, $d_H(x,x') = d_H(x-x',0) = \mathrm{wt}(x-x')$. We have:

$$A_k(C^\perp) = \sum_{\mathrm{wt}(z)=k}\mathbf{1}_{C^\perp}(z)^2 = \frac{q^{2n}}{|C|^2}\sum_{\mathrm{wt}(z)=k}\langle \mathbf{1}_C, \phi_z\rangle\langle \phi_z, \mathbf{1}_C\rangle = \frac{1}{|C|^2}\sum_{\mathrm{wt}(z)=k}\left|\sum_{x\in C}\phi_z(x)\right|^2.$$

Also,

$$|C|\sum_{w=0}^{n}A_w(C)K_k^{n,q}(w) = \sum_{x,y\in C}P_k(d_H(x,y)) = \sum_{x,y\in C}E_k(x,y) = \sum_{\mathrm{wt}(z)=k}\sum_{x,y\in C}\phi_z(x)\overline{\phi_z(y)} = \sum_{\mathrm{wt}(z)=k}\left|\sum_{x\in C}\phi_z(x)\right|^2.$$

As a result, we reach to the theorem, which has an introduction and generalization in [Sim95]:

Theorem 4.5 (MacWilliam's Identities)**.**

$$A_k(C^\perp) = \frac{1}{|C|}\sum_{w=0}^{n} A_w(C)K_k^{n,q}(w)$$

Remark 4.6. If we let $W_C(x,y) = \sum_{w=0}^{n} A_w(C)x^w y^{n-w}$, the identities above is represented in a more concise way:

$$W_{C^\perp}(x,y) = \frac{1}{|C|}W_C(y-x, y+(q-1)x).$$

◊

Thus we give the Delsarte inequalities:

$$\sum_{w=0}^{n} A_w(C)K_k^{n,q}(w) \geq 0, \qquad k = 0,\dots,n.$$

These are the linear constraints used in the LP bound. We have the linear program

$$\begin{aligned}
\max_{A_0,A_1,\dots,A_n} \quad & \sum_{t=0}^{n} A_t \\
\text{subject to} \quad & A_0 = 1, \\
& A_t = 0, \quad t = 1,\dots,\delta-1, \\
& A_t \geq 0, \quad t = \delta,\dots,n, \\
& \sum_{t=0}^{n} A_t K_k^{n,q}(t) \geq 0, \quad k = 0,1,\dots,n.
\end{aligned}$$

The dual program is

$$\begin{aligned}
m(X,\delta) = \min_{C_0,C_1,\dots,C_n} \quad & 1 + \sum_{k=0}^{n} C_k K_k^{n,q}(0) \\
\text{subject to} \quad & \sum_{k=0}^{n} C_k K_k^{n,q}(t) \leq -1, \quad t = \delta,\dots,n, \\
& C_k \geq 0, \quad t = 0,\dots,n.
\end{aligned}$$

which identifies the result from SDP (here every $d_H(x,x) = 0$), which is shown in Eq. (2) when the number of irreducible subspaces n is finite.

4.2 Kissing Number: Unit Sphere S^{n-1}

In this subsection, we consider $G = O_n = \{A : A \in \mathrm{GL}_n(\mathbb{R}), A^T A = I_n\}$ acting on S^{n-1} by $x \mapsto Ax$. For any fixed x_0, the stablizer $H \cong O_{n-1}$.

These representations are realized concretely by spherical harmonics. Let Harm_k^n denote the space of harmonic polynomials on $\mathbb{R}^n$ that are homogeneous of degree k. The restriction of such polynomials to S^{n-1} yields finite-dimensional subspaces

$$H_k^n = \left\{ f \in \mathbb{C}[X_1,\dots,X_n]_k, \Delta f = 0, \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \right\}\Bigg|_{S^{n-1}},$$

which are invariant and irreducible under O_n. Indeed, the Laplace operator Δ commutes with the orthogonal group, so Harm_k^n is a G-submodule, and it is known to be irreducible with highest weight $(k,0,\dots,0)$.

The Peter–Weyl decomposition takes the explicit form

$$\mathcal{C}(S^{n-1}) = \bigoplus_{k=0}^{\infty} H_k^n,$$

with $H_k^n \cong \mathrm{Harm}_k^n$ as O_n-modules. The direct sum is orthogonal with respect to the G-invariant inner product

$$\langle f, g\rangle = \frac{1}{\omega_n}\int_{S^{n-1}} f(x)\overline{g(x)}\, d\omega(x),$$

where ω_n is the surface measure of S^{n-1}.

The dimension of H_k^n is

$$h_k^n := \dim H_k^n = \binom{n+k-1}{k} - \binom{n+k-3}{k-2},$$

for $k \geq 2$, with $\dim H_0^n = 1$, $\dim H_1^n = n$.

Since S^{n-1} is 2-point homogeneous under O_n, each isotypic component H_k^n has multiplicity one. The associated zonal function corresponding to H_k^n depends only on the inner product $t = x \cdot y$. It is given, up to normalization, by the Gegenbauer polynomial $P_k^n(t)$ with parameter $\frac{n}{2} - 1$:

$$E_k(x, y) = h_k^n P_k^n(x \cdot y).$$

The orthogonality relations for these polynomials reflect the orthogonality of different H_k^n:

$$\int_{-1}^{1} P_k^n(t)P_\ell^n(t)(1-t^2)^{\frac{n-3}{2}}\, dt = 0, \quad k \neq \ell.$$

In $n = 3$, they are the well-known Legendre polynomials.

Thus, the Peter–Weyl decomposition of $\mathcal{C}(S^{n-1})$ under O_n is precisely the Fourier expansion in spherical harmonics.

4.2.1 Kissing Number Bound

Kissing number associated to dimension n is a maximal number of non-overlapping unit spheres that can touch a given central unit sphere in n-dimensional Euclidean space, denoted by τ_n. It is trivial that $\tau_1 = 2, \tau_2 = 6$. From high-school chemistry we also know $\tau_3 \geq 12$.

Then let us recall Theorem 4.4: Let $F(t) = b + \sum_{k=0}^{d} f_k P_k(t)$. If $f_k \geq 0$ for all $0 \leq k \leq d$ and $b > 0$, and if $F(t) \leq 0$ for all $t \in D_{\geq\delta}$, then

$$A(X, \delta) \leq \frac{F(e)}{b}.$$

Here we have $d(x, y) = x^T y$ so $e = d(x, x) = 1$. Let $F(x) = 1 + \sum_{k\geq 0} f_k P_k(t)$, if all $f_k \geq 0$ and $F(t) \leq 0$ for all $t \in D_{\geq\delta}$, then $A(X, \delta) \leq F(1)$. The $D_\delta = [-1, \frac{1}{2}]$ for there are no 2 points that has an angle of less than $\frac{\pi}{3}$. Take $F(t) = (t - 1/2)t^2(t + 1/2)^2(t + 1)$, and we can check

$$\frac{320}{3}F = \frac{1}{h_0^8}P_0 + \frac{16}{7h_1^8}P_1 + \frac{200}{63h_2^8}P_2 + \frac{832}{231h_3^8}P_3 + \dots$$

Recall $h_0^n = 1$. Thus b can be maximized to $\frac{3}{320}$ (because $P_0(t) = 1$, so $f_0 + b$ is fixed and we can have $f_0 = 0$). As a result, $A(X, \delta) \leq \frac{320F(1)}{3} = 240$.

There is a construction of the E_8 lattice:

$$\left\{(x_1, \dots, x_8) : \sum_{i=1}^{8} x_i \equiv 0 \pmod 2 \text{ and } \sum_{i=1}^{8} x_i^2 = 2\right\}.$$

The solution set is a set of cardinality 240, and every pair of vectors has the angle between them $\geq \frac{\pi}{3}$. Here we list them and count:

1. There are 2 ± 1 entries: $4 \times \binom{8}{2} = 112$;
2. All entries are from $\pm\frac{1}{2}$, and there are even positives: $\frac{1}{2} \times 256 = 128$.

and we can check that every inner product of different vectors is no more than 1. Thus we reached to the goal that $\tau_8 = 240$.

5 Miscellaneous

1. All scheduled lectures attended, with additionally $\sim$5 hours per week for reading materials. Topics include representation theory and topological groups. So maybe $16 \times 8 = 128$ **hours** in total.

2. The final report is finished in **one week**, but reading takes more time (including researching other materials that are not listed in the references) and begins in week 16. So I would estimate the total time of the report as 50 **hours** approximately.

3. I included no main theorems in this report, so it was more like a reading report than a paper. Maybe the highlight is that my report is clear about the main ideas for beginners to grasp throughout the reading, because I reorganized the content and focused more on "why this?".

References

[Bac10] Christine Bachoc. *Semidefinite programming, harmonic analysis and coding theory*. 2010. arXiv: 0909.4767 [cs.IT]. URL: https://arxiv.org/abs/0909.4767.

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